International Journal of Economics and Financial Issues

ISSN: 2146-4138

available at http: www.econjournals.com

International Journal of Economics and Financial Issues, 2025, 15(4), 343-355.

Enhancing Portfolio Asset Allocation Efficiency using Blended Covariance Matrices

Ruwan Pienaar[1], Gary van Vuuren[2,3]*

[1]School of Economics, University of Cape Town, Cape Town, South Africa. [2]Centre for Business Mathematics and Informatics, North-West University, Potchefstroom, South Africa, [3]National Institute for Theoretical and Computational Sciences, Pretoria, 0001, South Africa. *Email: 12001333@nwu.ac.za

Received: 10 December 2024 **Accepted:** 17 May 2025 **DOI:** https://doi.org/10.32479/ijefi.18883

ABSTRACT

The Mahalanobis distance (MD) is employed as a threshold for outlier identification in the construction of a blended covariance matrix, aimed at enhancing portfolio allocation and risk management during periods of elevated market volatility. During such times, sample covariance matrices often become unstable, resulting in unrealistic and unconstrained efficient frontiers. This study addresses the inadequacy of the sample covariance matrix for portfolio optimisation under turbulent conditions, where estimation errors can significantly undermine investment performance. By comparing the performance of portfolios constructed using both unaltered covariance matrices and those incorporating a blended approach informed by MD, the analysis demonstrates that the latter method substantially reduces portfolio volatility and improves risk management capabilities. The study finds that even with reduced risk, the phenomenon of an inverted efficient frontier persists during volatile periods, indicating a need for further research. Stabilised covariance matrices offer practical benefits for risk management and asset allocation in such environments. By examining the evolution of covariance matrix stability across both calm and turbulent market conditions, this research contributes to the understanding of covariance instability and highlights the value of the blended approach. The persistence of the inverted frontier underscores the enduring challenges in effective portfolio optimisation.

Keywords: Mahalanobis Distance, Efficient Frontier, Blended Covariance Matrix
JEL Classifications: C3, C38, C5, C53

1. INTRODUCTION

Portfolio construction has been significantly influenced by Markowitz (1952) who introduced mean-variance optimisation as a cornerstone method for asset selection. The idea is based on investors optimising portfolios to achieve an optimal trade-off between expected return and risk. This method relies on two fundamental components; the expected (excess) return for each stock, which indicates a portfolio manager's capacity to forecast future price movements, and the covariance matrix of asset returns, which illustrate the relationship between assets and act as a necessary tool for efficient risk management.

The anticipated returns and covariance matrix governing the portfolio behaviour are unknown and need to be estimated. Traditionally, the statistical methodology entails gathering historical stock returns and creating a sample covariance matrix. Bawa and Brown (1979) provide a comprehensive overview of the challenges associated with portfolio optimisation using historical asset return characteristics. Green et al. (2013) enumerate over 300 papers dedicated to the first estimation problem, the vector of expected returns, whereas comparatively less attention has been given to the covariance matrix.

The construction of the sample covariance matrix has been a topic of debate in the literature, with discussions centred on determining the most accurate method of calculating this matrix as an input into financial models. The sample covariance matrix has been shown to be an inadequate tool for portfolio optimisation due to a variety of factors. The presence of estimation error can undermine the desirable properties of the chosen investment portfolio, a phenomenon known as the estimation risk problem in portfolio selection. This challenge, illustrated by Michaud (1989), is exacerbated when the sample covariance matrix contains significant estimation errors, especially when the number of historical return observations per stock is close to or less than the number of stocks in the portfolio, or when the portfolio contains highly correlated stocks. The challenge pointed out by Ledoit and Wolf (2022) was described as the curse of dimensionality. This problem becomes particularly noticeable when the matrix dimension is larger than the sample size. In such cases, the sample covariance matrix becomes singular, meaning it lacks full rank. This issue persists even in other situations unless the matrix dimension is relatively small compared to the sample size, frequently arising in the financial market when the number of investment securities in the portfolio increases faster than the observed time (Dendramis et al., 2021; Raninen et al., 2022; Samal et al., 2021).

In addressing the challenge of estimating the covariance matrix for portfolio optimisation, researchers and practitioners have proposed various approaches. One such method involves the single-index (SI) model, as introduced by Sharpe (1964), where the covariance matrix is calculated for portfolio optimisation.[1] Sharpe (1964) was the first to demonstrate how an index model could potentially simplify the portfolio construction problem. Senneret et al. (2016) demonstrated that the SI approach, requiring the estimation of only $2N + 1$ parameters, outperforms the standard approach, which necessitates estimating $N(N+1)/2$ parameters. Despite these advancements, both standard and SI approaches remain susceptible to estimation errors, particularly in the use of the sample mean returns vector. Elton et al. (2006) set out the constant correlation (CC) model, assuming all stocks share the same correlation structure, to construct the covariance matrix. Forecasts of future correlation structures using the CC model had differences that were almost always statistically significant at the 0.05 level compared to the SI model and multi-factor models. Differences in portfolio performance were also economically significant, with the CC model often leading to an increase in returns of about 25% (Elton et al., 2006). However, the CC model faces challenges in estimating large-dimensional covariance matrices for portfolio selection. Another innovative approach, introduced by Ledoit and Wolf (2004), is the shrinkage method. This technique combines the sample covariance matrix with a stable structure shrinkage target matrix through a weighted combination, proving effective in portfolio selection, particularly in high-dimensional cases.

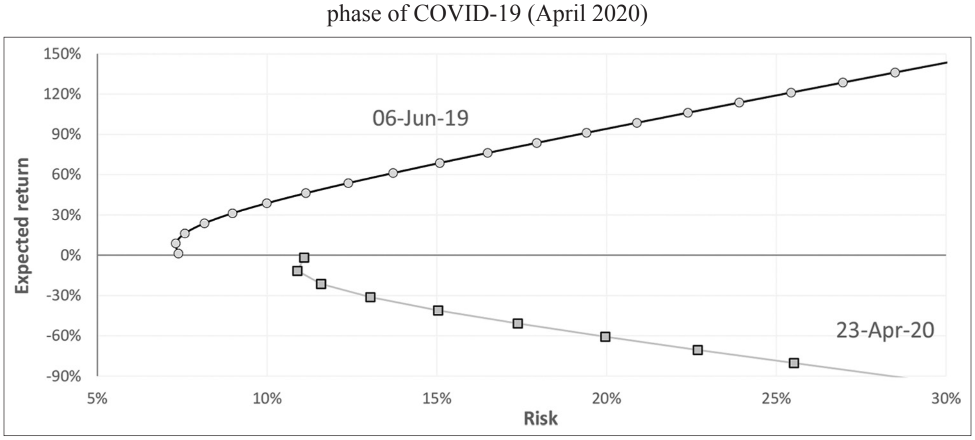
Extensive research has been conducted in addressing the dimensionality challenge, yielding several effective methods (see for example Ke et al., 2022). In this paper a newfound issue is tackled with a primary focus on the input covariance matrix. Volatile markets often trigger significant changes in key portfolio risk metrics like standard deviation, Value at Risk (VaR), correlations and asset covariances. Past relationships between assets break down, measurements become less precise and overall stability is compromised (Pál, 2022). This flow through to the sample covariance matrix of a portfolio which describes the strength of the relationships between assets. This overall instability creates a difficult and unrealistic task for portfolio construction and optimisation during these periods (Ke et al., 2022).

Traditional portfolio construction techniques, such as the efficient frontier, come under stress during volatile market periods (Fabozzi, Gupta, and Markowitz, 2002). When an unconstrained portfolio faces volatile market conditions, an unusual phenomenon arises. The sample covariance matrix becomes unstable, resulting in an inverted efficient frontier (Figure 1), a technically implausible outcome (Chan et al., 1999; Cohen and Pogue, 1967; Jobson and Korkie, 1981a; Ledoit and Wolf, 2004 and 2022).

The stability must be "restored" in a way such that rational portfolio construction can still take place during these periods. The blended Mahalanobis method introduced by Chow et al. (1999) will be incorporated to address this problem. This approach, which blends two covariance matrices (an inside-sample covariance

1 The Single Index Model is an extension of the Capital Asset Pricing Model. Developed by Sharpe (1964), it is used primarily to capture the stylised fact that the returns of many assets share a common component, namely exposure to a common source. The model assumes that each asset's return is linearly related to the return of a single common factor, usually an equity market index.

Figure 1: "Normal" efficient frontier when market conditions were non-turbulent (June 2019) AND during the turbulent initial phase of COVID-19 (April 2020)



matrix and an outlier-sample covariance matrix) which derives a new adjusted blended covariance matrix, which may be used as an input for portfolio construction and relevant risk management techniques) is a popular approach (Hunjra et al., 2020). The Mahalanobis distance (MD), which was originally introduced by Mahalanobis (1936) to analyse human skulls, is applied by Chow et al. (1999) in the financial context and was used to measure financial turbulence and identify multivariate outliers. This approach will be incorporated to address the stability issue of a sample covariance matrix during turbulent periods. In this paper we propose a solution to address the covariance instability issue and offer an alternative approach – the use of dynamic, blended, covariance matrices – during volatile periods.

1.1. Efficient Frontier

Markowitz (1952) is best known for his contributions to modern portfolio theory, such as the efficient market hypothesis (EMH) and the efficient frontier. According to the efficient market hypothesis, financial markets are efficient and reflect all available information about assets, resulting in prices that always reflect all information (Smith, Elton and Gruber, 1982). In essence, this theory contends that it is impossible to consistently outperform the market through stock picking or market timing because assets prices already include and reflect all relevant information. In the context of fund management, the Markowitz (1952) framework establishes the relationship between expected portfolio returns and the variance of those returns given a universe of investable assets. This relationship produces the well-known efficient frontier; the parabolic delineation in mean return/variance space. This parabolic delineation denotes a set of portfolios, where the optimal portfolio consists of the highest risk-adjusted return portfolio, maximising returns while simultaneously minimising risk. This frontier is also based on the concept of diversification, in which investors can achieve higher risk-adjusted returns by holding a diverse portfolio of assets (Markowitz, 1952).

Numerous researchers and practitioners have extended and applied the concept of the efficient frontier in various contexts. The work of Sharpe (1964) on the SI model introduced further sophistication to portfolio optimisation methods, utilising the efficient frontier to balance risk and return. Jensen, Black and Fischer (1972) introduced the Capital Asset Pricing Model (CAPM) which refines the efficient frontier by incorporating risk-free assets into portfolio optimisation, providing valuable insights into asset pricing and portfolio management.

Elton et al. (2006) explored the CC model, assuming a uniform correlation across all assets, which proved to be more accurate in forecasting future covariance matrices as an input for the efficient frontier compared to the traditional sample covariance matrix.

The mean variance optimiser of the efficient frontier reduces risk compared to the diversification rule by allocating higher weights to lower risk stocks and incorporating hedging relationships among stocks (Jobson and Korkie, 1980). Despite its theoretical appeal, mean variance optimisation has faced difficulties to translate this theoretical concept into practical investment strategies. According to Brandt (2010), mean variance portfolio optimisation frequently underperforms, rather than outperforms, simple diversification strategies such as equal weighting. This observation aligns with findings from DeMiguel et al. (2009, 2011) and Jobson and Korkie (1981b). Extreme holdings in a small number of stocks, parameter instability, hypersensitivity to changes in data, and inadequate out-of-sample reliability have all been often raised problems (Green and Hollifield, 1992). Steinbach (2001) argued that the subject is so complex that Markowitz's (1952) research raised more questions than it did answers, which led to a huge quantity of related research being conducted. Kan and Zhou (2007) show that when the ratio of the number of stocks to historical return observations is not small enough, estimation errors in return predictions and covariances lead to significant interactive effects, resulting in unstable and unreliable optimal weighting solutions.

Relying on historical data to estimate asset returns and covariance matrices may not accurately reflect future market conditions, particularly during times of high volatility. The efficient frontier model is also based on assumptions such as the normal distribution of asset returns and static correlations between assets, which may not be valid in real-world scenarios. Practitioners and theorists have found concerns regarding the efficient frontier. Practitioners and theorists have raised several concerns regarding the efficient frontier and its practical application in portfolio optimisation. DeMiguel et al. (2009) find that equal-weight portfolios often outperform the efficient frontier in terms of risk-adjusted returns, suggesting that estimation error can outweigh the benefits of optimal diversification. Baker and Nofsinger (2010) discuss how behavioural biases can affect the practical application of the efficient frontier, as investor behaviour often deviates from the assumptions of rationality. Maher et al. (2011) document the transitory nature of "efficient" weights across asset classes, indicating that the frontier can change significantly over time. Michaud (1989, 2013) refers to the efficient frontier as "error maximising" and proposes resampling approaches to reduce the impact of estimation error, while acknowledging the frontier's limitations. Feng and Wang (2016) explore how sampling error impacts the construction of the efficient frontier, finding that it can lead to significant deviations from the true frontier.

The literature has also explored Markov regime-switching and its impact on efficient frontiers and asset allocation, typically focusing on regimes like bullish and bearish markets or varying interest rates (Barbosa and Pereira, 2018 and Chen et al., 2023), but it has not specifically addressed efficient frontier inversion.

In the exploration of improving portfolio stability amidst market volatility, the efficient frontier is used as a practical tool rather than a definitive recommendation in this paper. The blended Mahalanobis approach uses the efficient frontier framework to evaluate the effectiveness of the technique during volatile market conditions, due to the inversion of the efficient frontier that occurred. This study only focusses on the instability of the sample covariance matrix during turbulent financial periods. This efficient frontier instability (Ledoit and Wolf, 2004 and 2022) raises critical questions about the reliability of traditional portfolio optimisation approaches during volatile periods. The challenge lies in adapting these methodologies to address the unique dynamics that arise when markets experience these extreme fluctuations (Raninen et al., 2022).

1.2. Blended Mahalanobis Method

In turbulent markets asset returns tend to be more volatile and have increased correlations (Chow et al., 1999; Forbes and Rigobon, 1999 and Boyer et al., 1999). This means that investments that have traditionally provided diversification lose some of their effectiveness when they are most needed to mitigate risk. Chow et al. (1999) illustrated this assertion with the following example. Long-Term Capital Management's (LTCM), a prominent hedge fund known for its sophisticated quantitative trading strategies, suffered significant losses during the market turmoil caused by Russia's default in 1998. The devaluation of the Ruble, combined with the investors fleeing to safe assets, resulted in previously uncorrelated or negatively correlated assets moving in tandem or even becoming positively correlated. As a result, LTCM's diversified portfolio, which assumed stable correlations between assets, did not provide the anticipated risk mitigation. Their strategies were severely impacted and incurred considerable losses. This event emphasises the importance of re-evaluating and adapting diversification strategies in response to changing market dynamics and increased correlations during volatile periods.

One way to look at this is to differentiate between time-measured and event-measured observations. When returns over time are measured, there may be periods where there are no significant events affecting prices, resulting in returns that are essentially noise. There are also times when multiple significant events influence returns (Osborne and Overbay, 2004). Conventional risk-assessment techniques give equal weight to all time periods, regardless of activity or significance (De Nard, 2022). The more perceptive method could involve adjusting returns based on significant market events rather than time intervals. By prioritising events that influence market dynamics, we may be able to provide a more precise assessment of risk during times of turmoil. This includes the challenging task of modelling the intricate dynamics of market behaviour, including both normal market fluctuations and anomalies that occur in financial landscapes (Li et al., 2023). We recognise the significant disruption that anomalies, such as market crashes or unexpected economic events, can cause in financial systems. It is thus critical that these anomalies are incorporated into financial models (De Nard, 2022).

Outliers have a major impact on statistical inference. They increase error variance (Iglewicz and Hoaglin, 1993), reduce the power of statistical tests (Barnett and Lewis, 1994), and result in biased estimates (Gress et al., 2020). As a result, the process of detecting outliers is an important aspect of data analysis. Synonyms for the outlier detection process vary depending on the application, but some examples include anomaly detection, deviation detection, exception mining, fault detection in safety critical systems, credit card fraud detection, intrusion detection in cyber security (unauthorised access in computer networks), misuse detection, noise detection, and novelty detection (Li et al., 2023). Distance-based techniques for identifying outliers, such as the *k*-nearest neighbour (k_{NN}) algorithm, compute the nearest neighbours of a record using a suitable distance calculation metric, such as Euclidean distance, MD, or another measure of dissimilarity. For large data sets, the MD is more computationally demanding than the Euclidean distance because it must pass through all variables in the data set to calculate the underlying inter-correlation structure (Dendramis et al., 2021; Ghorbani, 2019).

The MD, which is known to be useful for identifying multivariate outliers or uncommon relationships between two or more variables, was first introduced in life science research by Mahalanobis (1936). More specifically, in anthropometry (the study of human body measurements). The MD is based on the location and scatter of a multivariate distribution. The metric measures the range of any observation's space from the centre of its distribution. Researchers use the MD for this reason: to analyse differences between populations and identify potential outliers. Statisticians recognised its potential for analysing any kind of multivariate data, leading to it becoming a widely used tool in statistics and machine learning, having applications in anomaly detection, pattern recognition and identification of weak signals in noisy environments (Aue et al., 2009).

Chow et al. (1999) advanced upon the MD and applied it in the field of finance, providing a more accurate representation of how a portfolio is expected to perform in volatile markets than traditional time-based methods. Chow et al. (1999) incorporated the MD to evaluate whether a data range is categorised as a stress-related outlier or not. The two datasets, which were filtered according to the MD (namely the inside-sample and outlier-sample) are used to construct the new blended covariance matrix. This new covariance matrix will be weighted towards the outlier-sample based on the chosen required confidence interval.

The method proposed by Chow et al. (1999) to construct a covariance matrix has been rarely applied in portfolio construction based on real financial stock market data (although recent by Sikalo et al. (2022) has renewed interest in Chow et al. (1999) work). Previous studies were more interested in using the MD technique as a filter for turbulent periods or in using the findings regarding correlation structure. Kritzman et al. (2012) identified financial turbulence according to the MD as an economic variable alongside inflation and economic growth to forecasts regime shifts using a Markov-switching model. Jacquier and Marcus (2001) were interested in the cross-country correlation's findings by Chow et al. (1999) to conduct a study on correlation structure changes in the financial markets, whereas Behmiri et al. (2016) conducted a study on conditional correlation between commodities in the futures market.

We propose incorporating the MD as a filter to identify outliers and the construction of a blended covariance matrix, comprising a blend of an outlier covariance matrix and a normal/stable covariance matrix. This blended covariance matrix will provide the input covariance matrix and act as a stable matrix during turbulent times. It is proposed that this method will act as a risk management tool and replace (where needed) the sample covariance matrix for portfolio assembly.

Modelling the behaviour of participants in financial markets, which involve intricate and routine but complex activities, poses numerous challenges. One significant barrier is that human participants in these markets can engage in both deliberate and careless behaviour, while also being vulnerable to unforeseen factors necessitating quick responses. The task is to accurately characterise and model markets' fluctuations while also accounting for any occasional deviations or unexpected events that may occur (Aue et al., 2009).

1.3. Financial Turbulence

Financial turbulence and stress related periods are described by Chow et al. (1999) as periods that are unusual given their historical behaviour which include decoupling of correlated assets and convergence of uncorrelated assets, not necessarily periods characterised only by low or negative returns. Chow et al. (1999) made use of the MD as the appropriate filter in detecting stressed events. Other respected studies addressed the problem of financial turbulence in a different manner. Bollerslev (1986) made use of time-varying volatility models, particularly GARCH (Generalised Autoregressive Conditional Heteroskedasticity) models. Recognising that volatility fluctuates in response to market conditions, Ang and Bekaert (2002) used Markov regime switching models, which involves multiple structures for characterising different time-series behaviours. Implied volatility was introduced by Mayhew (1995) and models which used a mixture of asset price jumps and diffusion (for those prices characterised by substantial and inconsistent changes) were introduced by Das and Uppal (2004).

Chow et al. (1999) statistical description of financial turbulence (for periods generally considered tumultuous) has two distinct advantages over the most used indicator of financial stress: implied volatility. Implied volatility in the market is the one standard deviation range potential price movements away from the underlying price over a given period. The difference between the blended approach and other approaches is described by Kritzman and Li (2010) whose approach allows for the estimation of turbulence across any set of assets, whereas implied volatility is typically only applicable to assets with liquid option markets. Kritzman and Li's (2010) measurement is thus not limited by the availability of options trading and can be applied more broadly across various asset classes. Kritzman and Li's (2010) approach not only considers the magnitude of individual asset returns but also captures the interactions among different combinations of assets, meaning that it provides a more comprehensive understanding of how assets interact with each other within a portfolio, thereby offering deeper insights into market dynamics beyond just the volatility of individual assets.

2. MATERIALS AND METHODS

2.1. Data

This paper takes the perspective of a South African investor. A random portfolio of 20 liquid JSE listed stocks are selected from different industries as seen in Table 1. The broad variety of industries are incorporated to display the broader market reactions during turbulent times, rather than just focusing on one industry. These 20 stocks are assembled into a portfolio, on which the results are based.

Table 1: Constituent South African shares used in the portfolio assembly and analysis

Sector	Subsector	Institution
Financial services	Banking and financial services	Absa
	Insurance services	OUTsurance
	Investment banking and financial services	RMB
	Financial services and insurance	Sanlam
	Investment holding company	Remgro
Telecommunications	Telecommunications services	MTN
Consumer services	Retail pharmacy	Dis-Chem Pharmacies
	Restaurant and food services	Famous Brands
	Grocery retail	Shoprite
	Food and beverage manufacturing	Tiger Brands
Real estate	Real estate investment trust	Growthpoint Properties
Mining	Gold mining	AngloGold Ashanti
	Platinum mining	Northam Platinum
	Energy and chemicals	Sasol
Education	Private education services	Curro
	Higher education services	Stadio
Transport and logistics	Logistics and shipping services	Grindrod
Industrial	Diversified industrial and manufacturing	KAP
Entertainment and leisure	Hospitality and entertainment	Sun International
Automotive and transport services	Financial services (focus on transport and logistics)	Transaction Capital

The data – sourced from Morningstar (2024) – comprise weekly total returns (including dividends) covering the period January 2018 to August 2023, capturing just over 5 years' worth of market data (about 300 observations). These 5 years were characterised by a range of economic conditions which will ensure rigid research on the techniques applied. South African investors experienced a low inflationary period of 3.0% in 2020 and high inflation of 6.9% in 2022 (the South African Reserve Bank dropped interest rates during COVID-19 and hiked them post COVID, to tame inflation). Increasingly severe domestic constraints caused GDP growth to slow to 1.9% in 2022 from 4.7% in 2021. Contributing factors were a fall in mining output, and a stagnation of manufacturing output, as load-shedding and transportation bottlenecks worsened (Overview, 2024). This period also consists of the pre-COVID-19 financial stability, the COVID-19 period, with significant turbulence and market upheaval, and finally, a post-COVID-19 phase of recovery. The time frame also incorporates recent major global events that have impacted financial markets, including the ongoing conflict stemming from the Russia-Ukraine war, which began in February 2022 with no resolution in sight of time of writing (May 2024).

There is a fair split of all 11 JSE industries represented in the portfolio. Stocks include Transaction Capital (which experienced an 68.8% decline from 13 March 2023 to 16 March 2023 following the SA taxi unrest) and Shoprite (a consumer staples stock, which experienced a 48.4% appreciation since July 2020 when it bottomed on the back of COVID-19).

Figure 2 shows the rebased index prices and daily volatility for the constructed portfolio from January 2018 to August 2023, with prices rebased from January 2018 to 100. In Figure 2, an equal weighting was assigned to each share to assemble a portfolio showing elevated volatility between January 2020 and July 2021, which was primarily driven by the COVID-19 pandemic's impact on global financial markets. Figure 3 shows the portfolio weekly returns over the same period.

2.2. Methodology

A typical portfolio comprising N assets is investigated. A weighted index, also comprising N assets, such as a value-weighted index, serves as the benchmark. All assets are included in the universe of assets from which the portfolio manager chooses. Excess returns are defined in relation to the benchmark weights. The following notation is useful in solving the optimisation problem:
***w*:** vector of portfolio weights
***y*:** vector of stock returns
$\mu = E(y)$**:** vector of expected asset returns
$e = \mu - w'\mu$**:** vector of expected asset excess returns
and $\sum$=: covariance matrix of asset returns.

The portfolio manager's task is to provide estimates for ***e*** the vector of expected excess returns, and for $\sum$, the covariance matrix of asset returns. This is usually undertaken using historical returns, as stated earlier. The minimum of $w'\sum w$ assigns each asset with the relevant weighting. The blending of two covariance matrices, inside-sample and outlier-sample, were used to provide an estimator of $\sum$. Returns on the N assets were tested and found to be stationary, random, and normally distributed using the historical weekly price data described in Section 2.1. The portfolio returns are thus multivariate normal distributed with mean-vector $\boldsymbol{\mu}$ and covariance-matrix$\sum$ (Fama, 1976).

2.2.1. Efficient frontier

Some definitions are required to establish the methodologies required for the frontier. These are listed below (Jorion, 2003). Net short sales are allowed. The investment problem is constrained by the requirement that the portfolio must be fully invested thus the total portfolio weights $\sum w = 1$. The following parameters are also defined using Merton's (1972) terminology:

$$a = \boldsymbol{\mu}'\sum^{-1}\mu, b = \boldsymbol{\mu}'\sum^{-1}, c = 1'\sum^{-1}1 \text{ and } d = a - \frac{b^2}{c}$$

where **1** is a vector of 1s.

The efficient frontier is delineated by two pivotal portfolios: the Minimum Variance (MV) portfolio, dedicated to minimising the variance, and the Tangent-to-the-Efficient-Set (TG) portfolio, focused on maximising the return-to-risk ratio. The TG portfolio is strategically positioned along a line drawn from the origin which passes through the global minimum variance portfolio and intersects the efficient frontier. Together, these portfolios

Figure 2: Time series of equal weighted constructed portfolio of 20 stocks, rebased to 100 in January 2018 (rebased prices) combined with the 52-week standard deviation (annual vola-tility) of the portfolio returns rolled forward

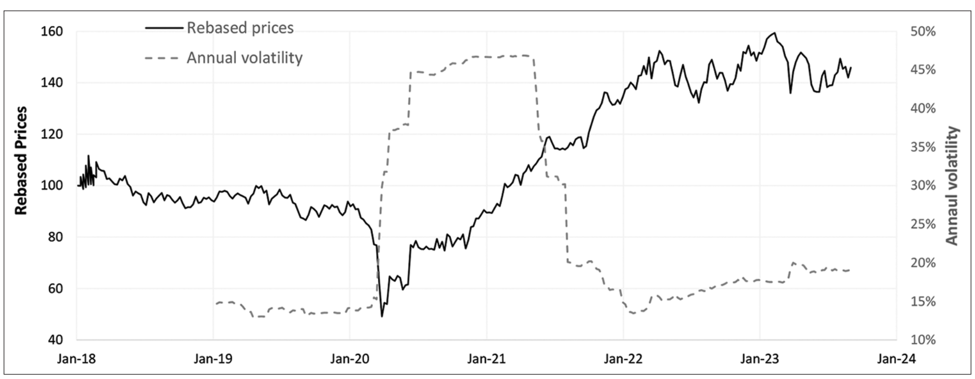


Figure 3: Portfolio weekly returns over the period under investigation

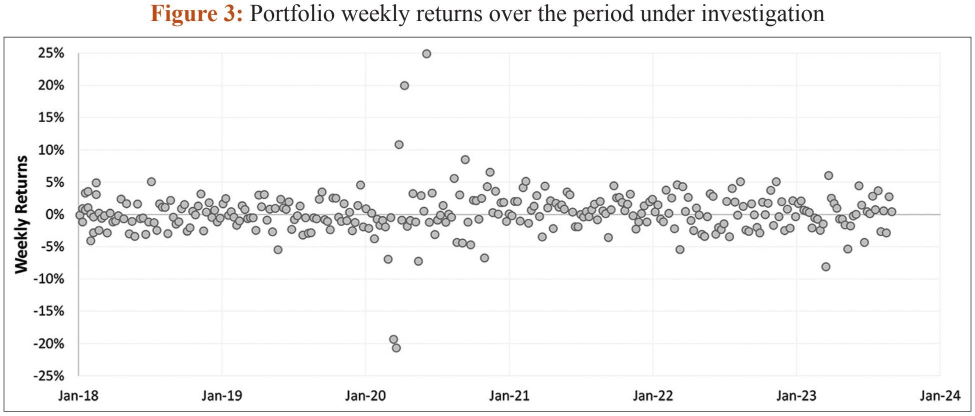
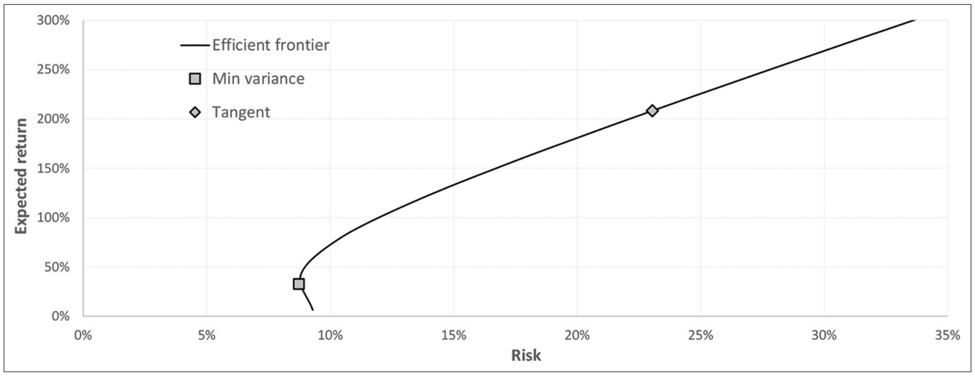


Figure 4: Efficient frontier including minimum variance portfolio and tangent portfolio. Based on portfolio in the week of 10 February 2022



contribute to a holistic definition of the efficient frontier: both are shown in Figure 4, which illustrates the efficient frontier. Roll (1992) demonstrated that the three parameters *a,b, and c* are interconnected with the means and variances of two pivotal portfolios (Table 2).

When the covariance matrix is positive definite, $a \geq 0$ and $c \geq 0$. The efficient set is meaningful when the expected return on the tangent portfolio is greater than the return on the minimum-variance portfolio, implying that $d \geq 0$. The efficient set, the TG portfolio which represents the best risk-return ratio, is a hyperbola in the (σ, μ) space with asymptotes of slope $\pm\sqrt{d}$.

2.2.2. Mahalanobis distance

Identifying an outlier in a returns series for a single asset refers to a return that falls outside of a predetermined confidence interval around the expected return, usually found in the tails of the distribution. A multivariate outlier poses a greater challenge in identification compared to a univariate outlier. It denotes a set of concurrent returns that are unusual for one or more reasons.

This could manifest in a variety of ways: for example, one of the returns in the set may deviate significantly from its average value, classifying the entire set of returns for that period as an outlier. Alternatively, a highly correlated pair of returns may show a significant difference in their individual returns, making the period unusual. As a result, a multivariate outlier can be caused by either exceptional performance of a single asset or exceptional interaction of multiple assets. Importantly, none of these assets must be individually unique to contribute to the outlier status of the overall set of returns. It is straightforward to visualise extreme Euclidean distances for univariate data, but the concept of "distance" must be modified to relate to extremeness for multivariate cases. Univariate outliers are statistically detected by converting raw data (x) into z-scores using the mean (μ) and standard deviation (σ) of the data using:

$$z = \frac{x-\mu}{\sigma} \tag{1}$$

Because the numerator and denominator use the same units, z-scores are dimensionless, "extreme" values are identified by the magnitude of z. For non-normal data, Chebyshev's Inequality guarantees a reasonable relationship between rarity and the

Table 2: Characteristics of portfolios MV and TG in terms of a, b and c

Portfolio	Expected return	Variance	Weights
MV	$E_{MV} = \frac{b}{c}$	$\sigma^2_{MV} = \frac{1}{c}$	$q_{MV} = \sum^{-1}\frac{1}{c}$
TG	$E_{TG} = \frac{a}{b}$	$\sigma^2_{TG} = \frac{a}{b^2}$	$q_{TG} = \sum^{-1}\frac{E}{b}$

unsigned distance of a point from the mean. Squaring Equation (1) and rewriting it facilitates an extension to multivariate data:

$$z^2 = (x-\mu)\,(\sigma^2)^{-1}\,(x-\mu) \tag{2}$$

$$z^2 = (x-\mu)^2\,\sigma^{-2})$$

Bivariate observations comprise two values, x and y, derived from two populations, X and Y. The pair of values can be represented by a column vector:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Assume that each population is normal distributed, but with different means, μ_x and μ_y and standard deviations, σ_x and σ_y, respectively. The centroid is:

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

Assume also that these variables are uncorrelated, meaning the covariance between them $\sigma_{xy} = 0$. The two types of distances (the Euclidean and the non-Euclidean distance) from point $P = [x,y]'$ to the centroid $\mu = [\mu_x,\mu_y]'$ are now examined.

When both X and Y distributions have zero means, the Euclidean distance from the point $[x,y]'$ to the origin follows the standard formula:

$$d^2 = x^2+y^2$$

The centroid is rarely situated at the origin because the means of the variables are not always zero, but rather at $[\mu_x,\mu_y]'$and the squared Euclidean distance of a point to the centroid is:

$$d^2 = (x-\mu_x)^2+(y-\mu_y)^2 \tag{4}$$

When dealing with variables of different units, such as miles per hour and fatalities, it becomes unclear how to interpret the resulting two-dimensional distance. Even when the variables share the same units, as in our study, Equation (4) remains sensitive to the chosen scale, leading to widely varying distance values across different units. This variability complicates the identification of extreme values, as Equation (4) disregards valuable information provided by the standard deviations. Standard deviations can also indicate the reliability of the measurement process; therefore, assigning equal weight to variables with differing reliability, as Equation (4) does, is inappropriate. The locus of all pairs of points $[x,y]$ equidistant from the centroid forms a circle, illustrating Equation (4)s equal weighting of the two variables in graphical terms.

This is where the non-Euclidean weighted distance to the centroid proves useful. The limitations of Equation (4) can be addressed by leveraging the rationale underlying the use of standard scores in Equation (1). Instead of directly incorporating $(x–\mu_x)$ and $(y–\mu_x)$ as components in a distance formula, a weight is first assigned to each difference based on the inverse of its associated standard deviation. In other words, weighted distances using z-scores are computed rather than simply relying on the differences from the means of the variables.

This approach addresses the concerns raised regarding the unweighted squared Euclidean distance. By utilising the dimensionless z-scores, the issue of mixing variables with different units is resolved. As z-scores are standardised measurements, the problem of variable scales is mitigated, allowing for the assessment of extremeness based on a function of those factors with greater variability receive less weight, thereby integrating the information conveyed by the magnitude of the standard deviation, implying:

$$d_w^2 = z_x^2 + z_y^2 \tag{5}$$

giving

$$d_w^2 = \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(x-\mu_y)^2}{\sigma_y^2} \tag{6}$$

Equation (6) can be used to determine the locus of all points having the same weighted distance from the centroid. Dividing both sides of Equation (6) by d_w^2:

$$1 = \frac{(x-\mu_x)^2}{\sigma_x^2 d_w^2} + \frac{(x-\mu_y)^2}{\sigma_y^2 d_w^2} \tag{7}$$

Equation (7) describes an ellipse centred at the centroid $\mu = [\mu_x, \mu_y]'$ with axes of length $\sigma_x d_w$ and $\sigma_y d_w$. Equation (7), being consistent with Equation (5), produces iso-weighted-distance contours. These contours suggest that points positioned along the same ellipse share comparable statistical "closeness" or "extremeness" from the centroid, even if their Euclidean distances vary. These ellipses are probability density contours, offering insights into the distribution of data around the centroid.

In matrix form we have

$$[p-\mu] = \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

For independent X and Y, the covariance matrix is

$$\sum = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$$

with inverse

$$\sum^{-1} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

and

$$d_w^2 = [(x-\mu_x),(y-\mu_y)] \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} \tag{8}$$

Equation (8) may be written

$$d_w^2 = [p-\mu]' \sum^{-1} [p-\mu] \tag{9}$$

which has structural and content similarities to Equation (2).

The structure of Equation (9) unveils a pattern that allows for the inclusion of additional variables, assuming all variables are uncorrelated. When extending beyond two variables, x and y are represented as $x_1, x_2\ldots$ and z_x and z_y become $z_1, z_2\ldots$more variables are introduced. Leveraging this new variable notation, the string of similarly patterned terms may be streamlined using an index and a summation sign, giving

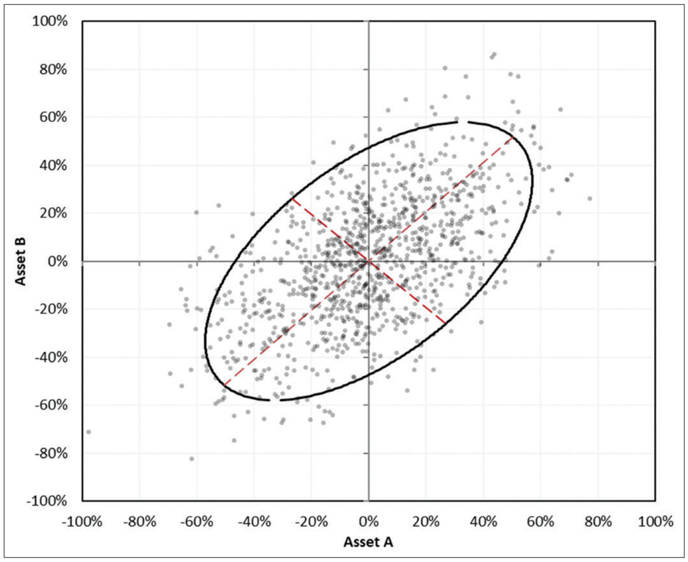
$$d_w^2 = \sum z_{x_i}^2 \tag{10}$$

Equation (10) is a sum of squared z-scores for the variables and is χ^2 distributed. This defines the Euclidean distance which fails to consider any relationship between assets, whereas the direction of asset returns relative to their means provides important information. For highly correlated assets, it is important when their returns deviate from their means in opposite directions. The squared MD accounts for this, providing a more accurate measure of deviation, particularly for correlated assets.

Thus far, the assumption of uncorrelated variables and zero covariances lacks realism when applied to real data. This constraint is also overcome by a Mahalanobis (1936) distance metric, which extends beyond the centroid to encompass the weighted distance between any two multivariate points. The method is modified by using ellipses rather than circles. The idea behind the MD is shown in Figure 5 using an example of the return series of two assets, A and B.

If $A=[A_1,A_2,\ldots,A_p]'$ and $B=[B_1,B_2,\ldots,B_p]'$ are multivariate observations drawn from a set of p variables with a 2 × 2

Figure 5: Scatterplot of two random assets' returns with unequal variances. A 90% confi-dence interval (solid oval) is also shown



covariance matrix S, the MD is defined as

$$d_m\,(A–B)^2=(A–B)'\,S^{-}1\ (A–B)$$

Each data point (A_1… and B_1…) reflects the return of assets A and B over a specific period, p. Assuming the ellipse represents the outlines of a bivariate-normal scatterplot with all data points along it being statistically equidistant from its centre. Acting as a tolerance boundary, the ellipse consists of a predetermined confidence region, with observations outside it classified as outliers. This centre denotes the average return of assets A and B.

Linear algebra is used when dealing with more than two return series. If the underlying distribution of the p random variables is exactly multivariate normal with a $p \times p$ covariance matrix $\sum$ and if $y = [y_1,y_2,\ldots,y_p]'$ has the constant value $\mu = [\mu_1,\mu_2,\ldots,\mu_p]'$ formed from the population means of the p random variables then the MD d_m of a particular multivariate data-point y from μ is:

$$d_m\,(y–\mu)^2 = (y–\mu)'\,\Sigma^{-1}\,(y–\mu)$$

Assigning a constant c to d_m defines a multidimensional ellipsoid centred at μ. The surface of this ellipsoid represents a probability density contour, and the probability linked with each probability contour following a χ^2 distribution with p degrees of freedom. The solid ellipsoid of values y satisfies

$$(y-\mu)'\sum^{-1}(y-\mu)\le \chi_p^2(\alpha) \qquad (11)$$

with the probability 1–α (Johnson and Wichern, 2007). This boundary separates turbulent and quiet observations, featuring graphical and numerical thresholds. Thus when $d_m(y)^2 \le \chi_p^2(\alpha)$ the observation is not classified as an outlier and when $d_m(y)^2 \ge \chi_p^2(\alpha)$ the observation is classified as an outlier.

The MD is advantageous due to its minimal reliance on distributional assumptions. Specifically, it serves as a suitable metric for measuring multivariate distance among elliptically distributed random variables, which are fully characterised by their location parameter μ and scatter matrix Σ (Mitchell and Krzanowski, 1985). The MD has several statistical properties. When asset returns follow a multivariate normal distribution, denoted as $r_t{\sim}N_n(\mu, \Sigma)$, the squared MD follows a χ^2 with n degrees of freedom. Under this assumption a specified confidence level is selected, and a boundary is calculated for each vector using the Chi-square distribution.

Chow et al. (1999) proposed a covariance matrix that effectively balance risk parameters derived from both stable periods, which are marked by low event activity, and outlier observations that represent turbulent, stressful periods.

The conventional mean-variance objective function is enhanced by incorporating the internal covariance matrix Σ_i and the outlying covariance matrix Σ_O. The two covariance matrices are classified by the boundary in Equation (11) and are assigned probabilities based on the chosen confidence interval, resulting in the new blended covariance matrix, replacing the sample covariance matrix. The sample covariance matrix is replaced by the new blended covariance matrix

$$\alpha\Sigma_i+(1–\alpha)\,\Sigma_O \qquad (12)$$

where α is the probability of falling within the sample (stable periods) and 1–α the probability of being an outlier. Asset returns are modelled as a discrete mixture of two distinct normal distributions. The χ^2 test assumes that all returns follow a single normal distribution, which introduces a minor contradiction. This contradiction is analogous to the common statistical scenario in which a test valid under a null hypothesis is used to reject that null hypothesis. It is unclear how this differs from the straightforward χ^2 test.

3. RESULTS

The initial 52 weeks (from January 2018 to November 2018) of the dataset were the starting point for the calculations. The same methodology was followed as in the initial 52 weeks and rolled forward on a weekly basis until August 2023. Subsequently, the in-sample, out-of-sample sample, and blended matrices were constructed for each period.

These methodologies and new blended covariance matrices were used in the construction of the efficient frontiers. The main metric observed and used in our analysis was the tangent portfolio, optimised for the best risk-adjusted portfolio for the 20 stocks.

The COVID-19 crisis significantly destabilized portfolio covariance matrices due to heightened market volatility and increased asset correlations (Díaz et al., 2022 and Li et al., 2022). Studies indicate that the pandemic led to a surge in uncertainty, causing traditional diversification strategies to falter as asset classes became more interlinked (Gharbi et al., 2022; Umar et al., 2021). Empirical research demonstrated that during the pandemic, the correlation among various asset classes surged, undermining the effectiveness of diversification and increasing overall portfolio risk. Behavioural finance factors, such as investor panic and herding behaviour, elevated market volatility, further destabilising covariance structures (Gharbi et al., 2022).

Based on the unconstrained nature of the portfolio, the efficient frontier experienced an inversion during the COVID-19 period. Figure 6 illustrates the inverted efficient frontier with the original sample covariance input and efficient frontier as well as the blended covariance matrix used as input rather than the sample covariance. Although the inversion is not corrected, the reduction in risk of the minimum variance and tangent portfolio can clearly be seen with the efficient frontier shifting to the left. Figures 7 and 8 display the risk and return metrics for the tangent portfolio based on an 80% confidence interval over the entire period. This tangent portfolio coordinates for all the 52-week efficient frontiers plotted, effectively the most optimal portfolio according to its risk-return trade-off.

The introduction of the blended covariance matrix significantly reduced the risk of the tangent portfolio, as evidenced by the lower standard deviation of returns observed during the period.

This risk reduction indicates that the blended covariance matrix has effectively reduced portfolio volatility, improving risk management capabilities. Making it a powerful tool for investors

Figure 6: Risk reduction with introduction of blended covariance matrix as input on efficient frontier at week of 19 March 2020

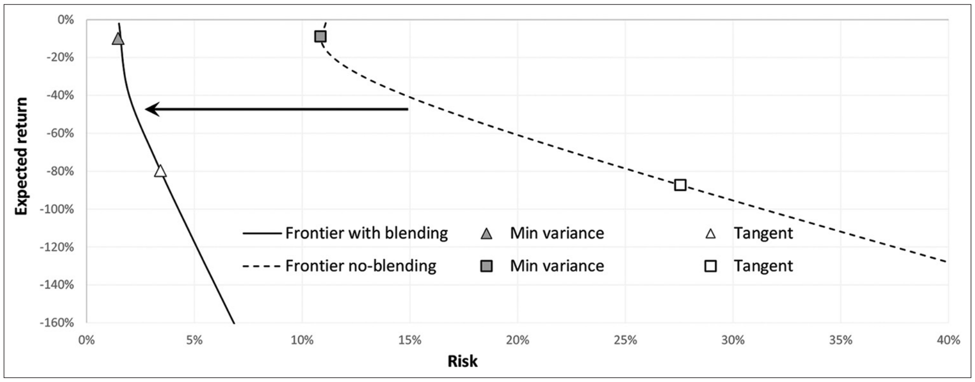


Figure 7: Comparison of original sample and blended covariance matrices via risk metric

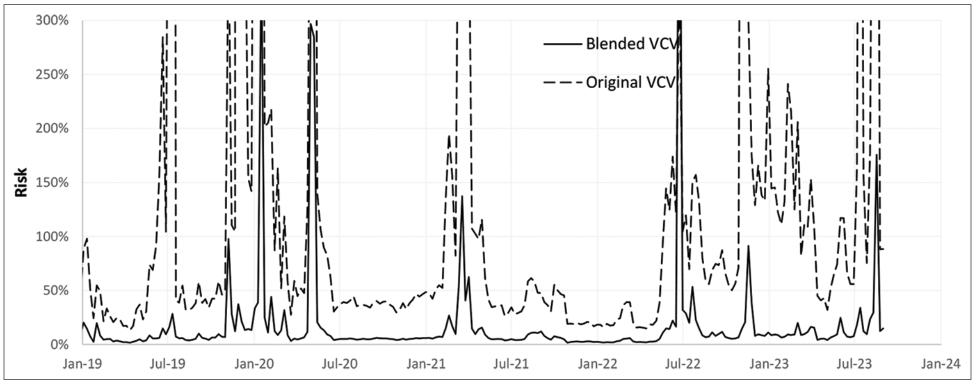


Figure 8: Comparison of original sample and blended covariance matrices via return metric

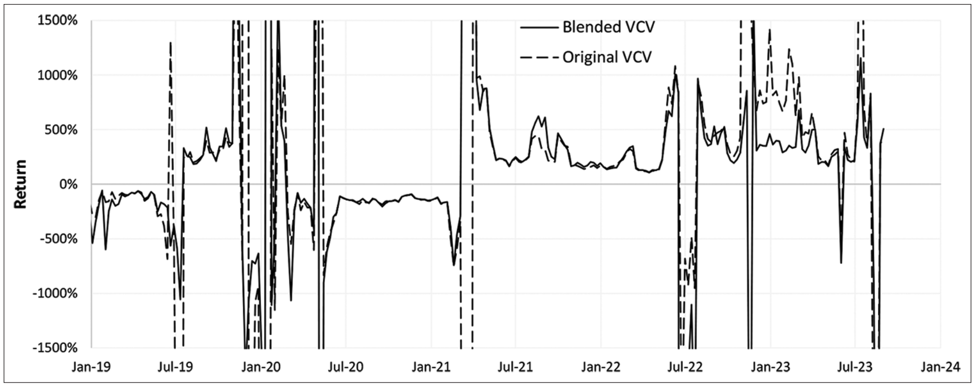
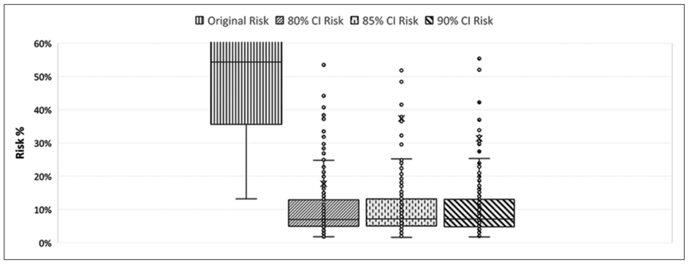


Figure 9: Box and Whiskers plot of different confidence intervals



seeking a balanced risk-return profile in their investment strategies and a more accurate representation of their covariance structure.

Other confidence intervals where also tested for efficacy. The 60-90% confidence intervals did not display any superior performance. Figure 9 displays the significant reduction in the risk measures using a box-and-whiskers plot. Each plot represents the tangent portfolio's standard deviation distribution of a specified confidence. The confidence interval exceeding 90% where to strict and did not account for any significant outliers, thus giving the same result as an 100% confidence interval. The return metric for the tangent portfolio where not affected at all by the change in the confidence intervals.

4. CONCLUSIONS

The research into the efficacy of blended covariance matrices, particularly using Chow et al.'s (1999) blended technique, has yielded valuable insights into portfolio optimisation and risk management strategies during periods of market volatility. Our findings highlight the blended covariance matrix approach's potential as a solution to challenges associated with the efficient frontier, portfolio building, and stress-testing investment strategies. The significant reduction in portfolio volatility observed after implementing the blended covariance matrix demonstrates its risk management effectiveness.

Despite the considerable reduction in portfolio volatility achieved through the implementation of the blended matrix, the study reveals that the inverted efficient frontier persists, suggesting that while the blended covariance matrix effectively mitigates risk, it does not fully rectify the distortion of the efficient frontier caused by extreme market conditions.

These findings encourage further investigation into alternative strategies or refinements to the blended covariance matrix approach. Addressing constraints and incorporating relevant industry-specific considerations will be critical in broadening the applicability of our findings. Investors frequently face constraints such as regulatory requirements, investment guidelines, or risk tolerance thresholds that limit the permissible allocation of assets. Ignoring these constraints and relying solely on an unconstrained portfolio may result in unrealistic asset weightings, potentially distorting portfolio performance metrics. Future work could restrict short selling and impose realistic – real world – asset allocation constraints. The introduction of a multi-asset portfolio would provide valuable insights, e.g., limiting investment weights to a range of assets (e.g., equity allocation restriction of 75%, offshore asset constraint of 45%, fixed income and cash >30% etc.). Such restrictions constrain the efficient frontier's risk and return profiles and prevent unrealistic short selling and asset allocation. Such analysis is more time consuming because no closed form solutions exist for ascertaining the efficient frontier constituent asset weights. Instead, estimation methods like Newton-Raphson or other iterative solution techniques are required. Including other assets classes could also provide deeper insights.

The empirical findings of DeMiguel et al. (2011) which suggest that non-theory-based portfolio diversification methods outperform more complex asset allocation techniques over time. The use of so-called sophisticated covariance estimation methods may not result in incremental gains for asset managers or investors. Equally weighted portfolio estimators, as advocated by Saghir et al. (2022) may serve as more prudent choices when designing investment strategies. Markov switching models which embrace efficient frontier inversion regimes could also prove to be a fruitful future pursuit.

REFERENCES

Ang, A., Bekaert, G. (2002), International asset allocation with regime shifts. The Review of Financial Studies, 15(4), 1137-1187.

Aue, A., Hörmann, S., Horváth, L., Reimherr, M. (2009), Break detection in the covariance structure of multivariate time series models. Annals of Statistics, 37(6B), 4046-4087.

Baker, H.K., Nofsinger, J.R. (2010), Behavioral finance: A new perspective on financial decision making. Financial Services Review, 19(1), 1-22.

Barbosa, A., Pereira, P.L.V. (2018), Asset allocation with Markovian regime switching: Efficient frontier and tangent portfolio with regime switching. Brazilian Review of Econometrics, 38(1), 97-116.

Barnett, V., Lewis, T. (1994), Outliers in Statistical Data. 3rd ed. New York: Wiley.

Bawa, V.S.S.J., Brown, R.W. (1979), Estimation Risk and Optimal Portfolio Choice, Studies in Bayesian Econometrics Bell Laboratories Series. New York: North Holland.

Behmiri, N.B., Manera, M., Nidycolini, M. (2016), Understanding Dynamic Conditional Correlations between Commodities Futures Markets. SSRN Electronic Journal. Available from: https://ideas.repec.org/p/fem/femwpa/2016.17.html#download

Bollerslev, T. (1986), Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3), 307-327.

Boyer, B., Gibson, M., Loretan, M. (1999), Volatility Risk and the Correlation of Asset Returns. Finance and Economics Discussion Series, 1999-31. Washington: Board of Governors of the Federal Reserve System.

Brandt, M.W. (2010), Portfolio choice problems. In: Handbook of Financial Econometrics: Tools and Techniques. Netherlands: Elsevier. p269-336.

Chan, L.K.C., Karceski, J., Lakonishok, J. (1999), The level and persistence of growth rates. Journal of Finance, 54(2), 659-682.

Chen, T., Liu, R., Wu, Z. (2023), Continuous-time mean-variance portfolio selection under non-Markovian regime-switching model with random horizon. Journal of Systems Science and Complexity. 36(3), 457-479.

Chow, G., Jacquier, E., Kritzman, M., Lowry, K. (1999), Optimal portfolios in good times and bad. Financial Analysts Journal, 55(3),

65-73.
Cohen, A., Pogue, T.F. (1967), A comparison of multi-factor models. Journal of Finance, 22(3), 527-542.
Das, S.R., Uppal, R. (2004), Systemic risk and international portfolio choice. The Journal of Finance, 59(6), 2809-2834.
De Nard, G. (2022), Oops! I shrunk the sample covariance matrix again: Blockbuster meets shrinkage. Journal of Financial Econometrics, 20(4), 569-611.
DeMiguel, V., Garlappi, L., Uppal, R. (2009), Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? Review of Financial Studies, 22(5), 1915-1953.
DeMiguel, V., Garlappi, L., Uppal, R. (2011), Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? In: Heuristics. Oxford: Oxford University Press. p644-664.
Dendramis, Y., Giraitis, L., Kapetanios, G. (2021), Estimation of time-varying covariance matrices for large datasets. Econometric Theory, 37(6), 1100-1134.
Díaz, F., Henríquez, P.A., Winkelried, D. (2022), Stock market volatility and the COVID-19 reproductive number. Research in International Business and Finance, 59, 101515.
Elton, E.J., Gruber, M.J., Spitzer, J. (2006), Improved estimates of correlation coefficients and their impact on optimum portfolios. European Financial Management, 12(3), 303-318.
Fabozzi, F.J., Gupta, F., Markowitz, H.M. (2002), The legacy of modern portfolio theory. The Journal of Investing, 11(3), 7-22.
Fama, E.F. (1976), Efficient capital markets: Reply. The Journal of Finance, 31(1), 143.
Feng, G., Wang, Y. (2016), The efficient frontier in the presence of sampling error. Journal of Asset Management, 17(5), 335-348.
Forbes, K.J., Rigobon, R. (1999), No contagion, only interdependence: Measuring stock market co-movements. Journal of Finance, 54(5), 2223-2261.
Gharbi, O., Trichili, Y., Boujelbene Abbes, M. (2022), Impact of the COVID-19 pandemic on the relationship between uncertainty factors, investor behavioral biases and the stock market reaction. Journal of Academic Finance, 13(1), 118-119.
Ghorbani, H. (2019), Mahalanobis Distance and its application for detecting multivariate outliers. Facta Universitatis. Series: Mathematics and Informatics, 34, 583.
Green, R.C., Hollifield, B. (1992), When will mean-variance efficient portfolios be well diversified? The Journal of Finance, 47(5), 1785-1809.
Green, J., Hand, J.R.M., Zhang, X.F. (2013), The preview of return predictive signals. Review of Accounting Studies, 18, 692-730.
Gress, T.W., Denvir, J., Shapiro, J.I. (2020), Effect of removing outliers on statistical inference: Implications to interpretation of experimental data in medical research. Journal of Medical Research, 12(4), 345-352.
Hunjra, A.I., Alawi, S.M., Colombage, S., Sahito, U., Hanif, M. (2020), Portfolio construction by using different risk models: A comparison among diverse economic scenarios. Risks, 8(3), 126-149.
Iglewicz, B., Hoaglin, D.C. (1993), How to Detect and Handle Outliers. New York: Sage Publications.
Jacquier, E., Marcus, A.J. (2001), Asset allocation models and market volatility. Financial Analysts Journal, 57(2), 16-30.
Jobson, J.D., Korkie, B. (1980), Estimation for Markowitz efficient portfolios. Journal of the American Statistical Association, 75(371), 544-554.
Jobson, J.D., Korkie, B. (1981a), Estimation for the mean and covariance of a portfolio. Journal of Financial and Quantitative Analysis, 16(3), 221-229.
Jobson, J.D., Korkie, B. (1981b), Putting Markowitz theory to work. The Journal of Portfolio Management, 7(4), 70-74.
Johnson, R.A., Wichern, D.W. (2007), Applied Multivariate Statistical Analysis. 6th ed. Upper Saddle River, NJ: Pearson.
Jorion, P. (2003), Portfolio optimization with tracking-error constraints. Financial Analysts Journal, 59(5), 70-82.
Kan, R., Zhou, G. (2007), Optimal portfolio choice with parameter uncertainty. Journal of Financial and Quantitative Analysis, 42(3), 621-656.
Ke, Y., Lian, H., Zhang, W. (2022), High-dimensional dynamic covariance matrices with homogeneous structure. Journal of Business and Economic Statistics, 40(1), 96-110.
Kritzman, M., Li, Y. (2010), Skulls, financial turbulence, and risk management. Financial Analysts Journal, 66(5), 30-41.
Kritzman, M., Page, S., Turkington, D. (2012), Regime shifts: Implications for dynamic strategies (corrected). Financial Analysts Journal, 68(3), 22-39.
Ledoit, O., Wolf, M. (2004), Honey, I shrunk the sample covariance matrix. The Journal of Portfolio Management, 30(4), 110-119.
Ledoit, O., Wolf, M. (2022), The power of (non-)linear shrinking: A review and guide to covariance matrix estimation. Journal of Financial Econometrics, 20(1), 187-218.
Li, L., Zhang, L., Zhu, Z. (2022), The impact of COVID-19 on portfolio investment: An empirical study on traditional and innovative asset classes. EUDL, 9(12), 232-276.
Li, Y.N., Li, D., Fryzlewicz, P. (2023), Detection of multiple structural breaks in large covariance matrices. Journal of Business and Economic Statistics, 41(3), 846-861.
Mahalanobis, P.C. (1936), On the generalized distance in statistics. Proceedings of the National Institute of Sciences (Calcutta), 2(1), 49-55.
Maher, M., White, H., Fry, P. (2011), Empirical problems using the efficient frontier to find optimal weights in asset classes. Journal of Accounting and Finance, 11(1), 1-12.
Markowitz, H. (1952), Portfolio selection. The Journal of Finance, 7(1), 77-91.
Mayhew, S. (1995), Implied volatility. Financial Analysts Journal, 51(4), 8-20.
Merton, R.C. (1972), An analytic derivation of the efficient portfolio frontier. Journal of Financial and Quantitative Analysis, 7(4), 1851.
Michaud, R.O. (1989), The Markowitz optimization enigma: Is 'optimized' optimal? Financial Analysts Journal, 45(1), 31-42.
Michaud, R.O. (2013), The efficient frontier: A new perspective. The Journal of Portfolio Management, 39(1), 12-20.
Mitchell, A.F.S., Krzanowski, W.J. (1985), The Mahalanobis distance and elliptic distributions. Biometrika, 72(2), 464.
Morningstar. (2024), Available from: https://www.morningstar.co./en-za/company [Last accessed 2023 Nov20].
Osborne, J.W., Overbay, A. (2004), The power of outliers (and why researchers should always check for them). Practical Assessment, Research, and Evaluation, 9(6), 1-9.
Overview, 2024 World Bank. Available from: https://www.worldbank.org/en/country/southafrica/overview [Last accessed on 2024 Apr 09].
Pál, L. (2022), Asset allocation strategies using covariance matrix estimators. Economics and Business, 10(5), 133-144.
Raninen, E., Tyler, D.E., Ollila, E. (2022), Linear pooling of sample covariance matrices. IEEE Transactions on Signal Processing, 70(2), 659-672.
Roll, R. (1992), A mean/variance analysis of tracking error. The Journal of Portfolio Management, 18(4), 13-22.
Saghir, A., Waheed, A., Malik, Q.A. (2022), Glittering covariance estimators: Sophisticated but worthless. Indian Journal of Economics

and Business, 21(2), 475-485.

Samal, A., Pharasi, H.K., Ramaia, S.J., Kannan, H., Saucan, E., Jost, J., Chakraborti, A. (2021), Network geometry and market instability. Royal Society Open Science, 8(2), 51-88.

Senneret, M., Malevergne, Y., Abry, P., Perrin, G., Jaffres, L. (2016), Covariance versus precision matrix estimation for efficient asset allocation. IEEE Journal of Selected Topics in Signal Processing, 10(6), 982-993.

Sharpe, W.F. (1964), Capital asset prices: A theory of market equilibrium under conditions of risk. The Journal of Finance, 19(3), 425-442.

Sikalo, M., Arnaut-Berilo, A., Zaimovic, A. (2022), Efficient asset allocation: Application of game theory-based model for superior performance. International Journal of Financial Studies, 10(1), 20-33.

Smith, K.V., Elton, E.J., Gruber, M.J. (1982), Modern portfolio theory and investment analysis. The Journal of Finance, 37(5), 1317.

Steinbach, M.C. (2001), Markowitz revisited: Mean-variance models in financial portfolio analysis. SIAM review. Society for Industrial and Applied Mathematics, 43(1), 31-85.

Umar, Z., Gubareva, M., Teplova, T. (2021), The impact of COVID-19 on commodity markets volatility: Analyzing time-frequency relations between commodity prices and coronavirus panic levels. Resources Policy, 73, 102164.